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On the Lipschitz Regularity of Minimizers of Anisotropic Functionals

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We prove a Lipschitz regularity result for minimizers of functionals of the calculus of variations of the form $\int_\Omega f(Du(x))\,dx$, where f is a continuous convex function from $\mathbb{R}^n$ into $[0, +\infty)$, not necessarily depending on the modulus of Du.

Key Words: minimizer; Lipschitz-regularity; convexity; general growth conditions; Δ_2-condition.

1. INTRODUCTION

In this paper we deal with regularity properties of local minimizers of functionals of the type

$$\int_\Omega f(Du)\,dx, \tag{1}$$

where Ω is a bounded open set in $\mathbb{R}^n$ (with $n \geq 2$), $u\colon \Omega \to \mathbb{R}$ is a weakly differentiable function, and $f\colon \mathbb{R}^n \to [0, +\infty)$ is a continuous convex function.

A function u from the class

$$W^{1,f}(\Omega) = \left\{ v \in W^{1,1}(\Omega)\colon \int_\Omega f(Dv)\,dx < +\infty \right\} \tag{2}$$

is said to be a local minimizer of the functional (1) if

$$\int_{\text{supp}(\varphi)} f(Du(x))\,dx \leq \int_{\text{supp}(\varphi)} f(Du(x) + D\varphi(x))\,dx,$$

for every $\varphi \in W^{1,1}(\Omega)$ such that $\text{supp}(\varphi) \subset\subset \Omega$.

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Very anisotropic functionals may have irregular minimizers, as shown by the examples contained in [16, 17, 21]. On the other hand, conditions on f ensuring regularity for minimizers are known. See for instance [2, 4–6, 8, 9, 13–15, 18, 22–24, 26–28, 30].

Our assumptions on f read as follows:

- f is even, namely

$$f(-\xi) = f(\xi) \tag{3}$$

for every $\xi \in \mathbb{R}^n$;

- f is coercive, namely

$$f(\xi) \geq c_0 |\xi|^2 \tag{4}$$

for some $c_0 > 0$ and every $\xi \in \mathbb{R}^n$;

- given

$$\lim_{|\xi| \to 0} \frac{f(\xi)}{|\xi|} = 0; \tag{5}$$

- there exists $p \geq 2$ such that

$$\langle Df(\xi), \lambda \rangle \leq p \frac{f(\xi)}{|\xi|} \, |\lambda|, \tag{6}$$

for every $\lambda \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^n$ at which f is differentiable;

- f is uniformly convex in $\mathbb{R}^n$, in the sense that there exists $\nu > 0$ such that for every $\xi, \lambda \in \mathbb{R}^n$

$$\frac{1}{2}[f(\xi) + f(\lambda)] \geq f\left(\frac{\xi + \lambda}{2}\right) + \nu \frac{f((\xi + \lambda)/2)}{|\xi|^2 + |\lambda|^2} |\xi - \lambda|^2. \tag{7}$$

Let us make a few comments about the above assumptions. Since f is convex and satisfies (3)–(5), then in particular it is a *generalized N-function* in the sense described in [29, 31, 33].

As far as condition (6) is concerned, it obviously implies that

$$\langle Df(\xi), \xi \rangle \leq p f(\xi) \qquad \forall\, \xi \in \mathbb{R}^n. \tag{8}$$

Inequality (8) is in turn equivalent to the so-called Δ_2-condition

$$f(2\xi) \leq \sigma f(\xi) \tag{9}$$

for some constant $\sigma \geq 2$ and every $\xi \in \mathbb{R}^n$. Indeed by (8) it easily follows that, for every $t \geq 1$,

$$\log\left(\frac{f(t\xi)}{f(\xi)}\right) = \int_1^t \frac{\langle Df(s\xi), \xi \rangle}{f(s\xi)} ds \leq p \int_1^t \frac{ds}{s} = p \log t,$$

whence

$$f(t\xi) \leq t^p f(\xi), \tag{10}$$

so that (9) holds with $\sigma = 2^p$.

Conversely, if (9) is fulfilled, then there exists a constant $p > 1$ such that

$$\langle Df(\xi), \xi\rangle \leq f(2\xi) - f(\xi) \leq (\sigma - 1)f(\xi), \tag{11}$$

owing to the convexity of f.

Note also that by (10)

$$f(\xi) \leq M|\xi|^p, \tag{12}$$

where $M = \max_{|\xi|=1} f(\xi)$.

The main result of the paper is the following:

THEOREM 1.1. *Under assumptions* (3)–(7), *every local minimizer u of functional* (1) *belongs to* $W^{2,2}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$. *More precisely, there exist positive constants $c = c(n, \nu, c_0, M, p)$, $\tilde{c} = \tilde{c}(n, \nu, c_0, M, p)$, and $\mu = \mu(n, p)$ such that*

$$\int_{B_{R/2}} |D^2u|^2\, dx \leq c\, R^{-2(2^p-2)} \int_{B_R} [1 + f(Du)]\, dx,$$

and

$$\sup_{B_{R/2}} |Du|^2 \leq \frac{\tilde{c}}{R^\mu} \int_{B_R} [1 + f(Du)]\, dx$$

for every $R > 0$ such that $B_R \subset\subset \Omega$.

Assumptions (3)–(7) allow us to consider functionals satisfying nonstandard growth conditions. Indeed, functionals whose integrand f is bounded from above and from below by the same function $A = A(\xi)$ are usually taken into account in the literature, under the assumption that A is a radial function from the class Δ_2 (see for instance [9, 18, 23]).

Here we do not assume any radial-symmetry on f. Moreover, although (12) holds, we will not impose any condition on the exponent p (see also [19, 23] for related results).

Another relevant feature of Theorem 1.1 is that it applies also to nonsmooth even functionals. An example of an integrand for which our statement holds is

$$f(\xi) = |\xi|^2 + \frac{|\xi|^p}{1 + |\xi|} \sum_{h=1}^{n} |\xi_h|.$$

Note that the partial derivatives $f_{\xi_i}(\xi)$ of this function are not continuous, whenever $\xi \not\equiv 0$ and $\xi_i = 0$ for some $1 \leq i \leq n$.

The proof of Theorem 1.1 goes through an approximation argument which is needed to overcome the lack of smoothness of the integrand f. Indeed we consider a sequence of functions F_k of class $C^2(\mathbb{R}^n)$, which converges to f uniformly on compact sets and satisfies conditions that are close to those imposed on f.

The construction of the sequence F_k is the task of Section 2 and is related to Lemma 2.4 of [11].

In Section 3, techniques from the standard theory of regularity are used to prove the twice weak differentiability and local Lipschitz regularity of minimizers of the functionals

$$\int_\Omega F_k(Dv)\,dx.$$

Finally, in Section 4, we conclude by passing to the limit as $k \to \infty$ in the estimates derived in Section 3.

2. PRELIMINARY RESULTS

In this section we present some technical result that will be used in the following.

First of all we recall some known features about the difference quotients of a function.

Let u be a measurable function on $\Omega \subset \mathbb{R}^n$ and let e_s be the sth coordinate unit vector in $\mathbb{R}^n$. Then for $h > 0$ the *difference quotient* of u in the direction e_s

$$\Delta_h^s u = \frac{u(x + he_s) - u(x)}{h}$$

is defined and measurable in $\Omega_h = \{x \in \Omega\colon d(x, \partial\Omega) < h\}$.

As it is well known, many properties of the function u in Ω hold as well for $\Delta_h^s u$ in Ω_h. When no confusion arises we will simply write $\Delta_h u$ to denote $\Delta_h^s u$.

The first result we want to prove is the following

LEMMA 2.1. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^+$ be a convex function. If $\Omega_0 \subset\subset \Omega$ and $h = (h_1, \dots, h_n) \in \mathbb{R}^n$ is such that $0 < |h| < h_0 = d(\Omega_0, \partial\Omega)$, then for every $v \in W^{1,f}_{\mathrm{loc}}(\Omega)$ we have*

$$\int_{\Omega_0} f(\Theta_h v)\,dx \le \int_\Omega f(Dv)\,dx,$$

where $\Theta_h u = (\Delta_{h_1}^1 u, \dots, \Delta_{h_n}^n u) \in \mathbb{R}^n$.

Proof. Since for $s = 1, \ldots, n$ we can write

$$(\Theta_h v)_s = \int_0^1 D_s v(x + th_s e_s)\, dt,$$

by means of the Jensen inequality we have

$$\begin{aligned}\int_{\Omega_0} f(\Theta_h v)\, dx &\le \int_{\Omega_0} \int_0^1 f(Dv(x+th))\, dt\, dx \\ &\le \int_0^1 \int_{\Omega_0 + th} f(Dv(y))\, dy\, dt \le \int_\Omega f(Dv)\, dx,\end{aligned}$$

since $\Omega_0 + th \subset\subset \Omega$. ∎

By the definition of (2) it is easy to deduce the corresponding definitions for the subspaces $W^{1,f}_{\mathrm{loc}}(\Omega)$ and $W^{1,f}_0(\Omega)$. Moreover we denote by $W^{1,f,\tau}(\Omega)$ the functions space defined by

$$\left\{ v \in W^{1,1}(\Omega) \colon \int_\Omega f(\tau D v)\, dx < +\infty \right\},$$

which reduces trivially to $W^{1,f}(\Omega)$ for $\tau = 1$.

In the next result we prove that for some $\tau > 0$ the difference quotient of $u \in W^{1,f}$ is a function of class $W^{1,f,\tau}_{\mathrm{loc}}$ and that its composition with a Lipschitz continuous function and the product by a regular function preserve this property.

LEMMA 2.2. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^+$ be convex and $v \in W^{1,f}(\Omega)$. Fix $h_0 > 0$ small enough and $h = (h_1, \ldots, h_s) \in \mathbb{R}^n$ such that $0 < |h| < h_0$. Then for every $s \in \{1, \ldots, n\}$ the following properties hold*

(i) $\Delta_{h_s} v \in W^{1,f,\tau_1}(\Omega_{h_0})$, *where* $\tau_1 = h_s/2$.

(ii) *If $\psi\colon \mathbb{R} \to \mathbb{R}$ is a Lipschitz-continuous increasing odd function* (*so that $\psi(0) = 0$ and*, *consequently*, $|\psi(t)| \le t\psi'(t)$), *then setting $K_1 = \|\psi\|_\infty + \|\psi'\|_\infty$ we have that $\psi(\Delta_{h_s} v) \in W^{1,f,\tau_2}(\Omega_0)$, where $\tau_2 = \tau_1/K_1$.*

(iii) *Let η be a function of class $C^1_0(\Omega)$ and ψ as in* (ii). *Then by setting $K_2 = \|\eta\|_\infty$ we have $\eta^2 \psi(\Delta_{h_s} v) \in W^{1,f,\tau_3}(\Omega_0)$, where $\tau_3 = \min\{\tau_2/(2K_2^2), (4K_1K_2)^{-1}\}$.*

Proof. Since by our assumptions f is convex, increasing in each direction, and even we have, by setting $\tau_1 = h_s/2$,

$$\begin{aligned} f(D(\tau_1 \Delta_{h_s} v)) &= f\left(\frac{1}{2} Dv(x + h_s e_s) - \frac{1}{2} Dv(x) \right) \\ &\le \frac{1}{2} f(Dv(x + h_s e_s)) + \frac{1}{2} f(Dv(x)) \end{aligned}$$

so that (i) follows.

Let us set $K_1 = \|\psi\|_\infty + \|\psi'\|_\infty$. Then by the chain rule for weakly differentiable functions we deduce that $D(\psi(\Delta_{h_s} v)) \le K_1 D(\Delta_{h_s} v)$ and then, if we set $\tau_2 = \tau_1/K_1$,

$$f(\tau_2 D(\psi(\Delta_{h_s} v))) \le f(\tau_1 D(\Delta_{h_s} v))$$

and the assertion follows by (i).

Now let $K_2 = \|\eta\|_\infty$. Since

$$D(\eta^2 \psi(\Delta_{h_s} v)) \le \eta^2 D(\psi(\Delta_{h_s} v)) + 2\eta D\eta \psi(\Delta_{h_s} v),$$

setting $\tau_3 = \min\{\tau_2/(2K_2^2), (4K_1K_2)^{-1}\}$ and using the convexity of f it follows that

$$\begin{aligned} f(\tau_3 D(\eta^2 \psi(\Delta_{h_s} v))) &\le f\left(\frac{\tau_2}{2} D\psi(\Delta_{h_s} v) + \frac{1}{2} D\eta\right) \\ &\le \frac{1}{2} f(\tau_2 D\psi(\Delta_{h_s} v)) + \frac{1}{2} f(D\eta) \end{aligned}$$

and then (iii) follows by (ii). ∎

We will also need the following result, in which we show that for a regular function, the uniform convexity condition (7) is equivalent to a condition that looks more familiar for twice differentiable convex functions:

LEMMA 2.3. *Let F be a convex function of class $C^2(\mathbb{R}^n)$. Then F satisfies* (7) *for every $\xi, \lambda \in \mathbb{R}^n$ if and only if the inequality*

$$\langle D^2 F(\xi)\lambda, \lambda\rangle \ge \tilde{\nu} \frac{F(\xi)}{|\xi|^2} |\lambda|^2 \tag{13}$$

holds, where $\tilde{\nu} = \tilde{\nu}(\nu)$.

Proof. Let us assume first that condition (7) holds for F.

For every $t > 0$ and $\xi, \lambda \in \mathbb{R}^n$ we have that

$$\begin{aligned} &\frac{1}{2}[F(\xi + t\lambda) - F(\xi)] + \frac{1}{2}[F(\xi - t\lambda) - F(\xi)] \\ &\qquad \ge \nu \frac{F(\xi)}{|\xi + t\lambda|^2 + |\xi - t\lambda|^2} |2t\lambda|^2. \end{aligned} \tag{14}$$

Since F is of class C^2, by applying the Taylor formula we have

$$F(\xi + t\lambda) = F(\xi) + t\langle DF(\xi), \lambda\rangle + \frac{t^2}{2}\langle D^2 F(\xi)\lambda, \lambda\rangle + o(t^2), \tag{15}$$

and

$$F(\xi - t\lambda) = F(\xi) - t\langle DF(\xi), \lambda\rangle + \frac{t^2}{2}\langle D^2 F(\xi)\lambda, \lambda\rangle + o(t^2) \tag{16}$$

so that, by adding up (15), (16) and by (14) we obtain

$$\frac{t^2}{2}\langle D^2F(\xi)\lambda, \lambda\rangle + o(t^2) \geq 4\nu t^2 \frac{F(\xi)}{|\xi + t\lambda|^2 + |\xi - t\lambda|^2}|\lambda|^2$$

and finally, dividing by t^2 and letting $t \to 0$ we easily get (13).

Conversely, let us assume that (13) holds. We set, for every $X, Y \in \mathbb{R}^n$,

$$h(t) = F(X + t(Y - X)).$$

Then we have

$$\begin{aligned} h'(t) &= \langle DF(X + t(Y - X)), Y - X\rangle, \\ h''(t) &= \langle D^2F(X + t(Y - X)) \cdot (Y - X), Y - X\rangle, \\ h(0) &= h(1) - h'(1) + \int_0^1 th''(t)\,dt \end{aligned} \tag{17}$$

and then by (17)

$$\begin{aligned} F(X) = F(Y) &- \langle DF(Y), Y - X\rangle \\ &+ \int_0^1 t\langle D^2F(X + t(Y - X)) \cdot (Y - X), Y - X\rangle\,dt. \end{aligned} \tag{18}$$

Now let $\xi, \lambda \in \mathbb{R}^n$. We consider (18), once with $X = \xi$ and then with $X = \lambda$; we add up the two equations and by (13) we easily get

$$\begin{aligned} F(\xi) + F(\lambda) = 2F(Y) &+ \int_0^1 t[\langle D^2F(\xi + t(Y - \xi)) \cdot (Y - \xi), Y - \xi\rangle \\ &+ \langle D^2F(\lambda + t(Y - \lambda)) \cdot (Y - \lambda), Y - \lambda\rangle]\,dt \\ \geq 2F(Y) &+ \tilde{\nu}\int_0^1 t\left[\frac{F(\xi + t(Y - \xi))}{|\xi + t(Y - \xi)|^2}|Y - \xi|^2\right. \\ &\left.+ \frac{F(\lambda + t(Y - \lambda))}{|\lambda + t(Y - \lambda)|^2}|Y - \lambda|^2\right]dt. \end{aligned}$$

Now we set $Y = \frac{\xi+\lambda}{2}$. Since in this case, as one can easily show, $|\xi + t \times (Y - \xi)|^2 \leq 2(|\xi|^2 + |\lambda|^2)$ and $|\lambda + t(Y - \lambda)|^2 \leq 2(|\xi|^2 + |\lambda|^2)$, we deduce that

$$\begin{aligned} F(\xi) + F(\lambda) \geq 2F\left(\frac{\xi + \lambda}{2}\right) + \frac{\tilde{\nu}|\xi - \lambda|^2}{8(|\xi|^2 + |\lambda|^2)}\int_0^1 t[F(\xi + t(Y - \xi)) \\ + F(\lambda + t(Y - \lambda))]\,dt. \end{aligned} \tag{19}$$

Now, if we set $g(t) = t[F(\xi + t(Y - \xi)) + F(\lambda + t(Y - \lambda))]$, we observe that g turns out to be a convex function for $t \in [0, 1]$ since

$$\begin{aligned} g''(t) = t[&\langle D^2F(\xi + t(Y - \xi)) \cdot (Y - \xi), Y - \xi\rangle \\ &+ \langle D^2F(\lambda + t(Y - \lambda)) \cdot (Y - \lambda), Y - \lambda\rangle] \geq 0 \end{aligned}$$

and then, by means of Jensen inequality and (19), we deduce

$$F(\xi)+F(\lambda)\geq 2F\left(\frac{\xi+\lambda}{2}\right)+\frac{\tilde{\nu}|\xi-\lambda|^2}{8(|\xi|^2+|\lambda|^2)}g\left(\frac{1}{2}\right).$$

Finally, by the definition of g and since F is convex,

$$\begin{aligned}g\left(\frac{1}{2}\right)&=\frac{1}{2}F\left(\xi+\frac{1}{2}(Y-\xi)\right)+\frac{1}{2}F\left(\lambda+\frac{1}{2}(Y-\lambda)\right)\\&\geq F\left(\frac{\xi}{2}+\frac{Y-\xi}{4}+\frac{\lambda}{2}+\frac{Y-\lambda}{4}\right)\\&=F\left(\frac{\xi+\lambda}{2}\right).\end{aligned}$$

Hence we can conclude that

$$\frac{1}{2}[F(\xi)+F(\lambda)]\geq F\left(\frac{\xi+\lambda}{2}\right)+\frac{\tilde{\nu}}{16}\frac{F\left(\frac{\xi+\lambda}{2}\right)}{|\xi|^2+|\lambda|^2}|\xi-\lambda|^2.$$ ∎

Let $\rho(\xi)=\rho(|\xi|)$ be a standard radially symmetric mollifier, such that $\operatorname{supp}(\rho)=B_1(0)$, $\int_{B_1}\rho(\xi)\,d\xi=1$. Moreover, for $0<\varepsilon\leq 1/4$, let us introduce a sequence of regularizations of f,

$$f^{\varepsilon}(\xi)=(\rho_{\varepsilon}*f)(\xi)=\int_{B_1}\rho(\omega)f(\xi+\varepsilon\omega)\,d\omega, \tag{20}$$

where $\rho_{\varepsilon}(\xi)=\varepsilon^{-n}\rho(|\xi|/\varepsilon)$.

In the next result we prove that such functions are characterized by the properties discussed in Section 1.

LEMMA 2.4. *The functions f^{ε} defined by* (20) *converge to f as $\varepsilon\to 0$, uniformly on compact subsets of $\mathbb{R}^n$* (*as it is well known*) *and satisfy the conditions*

$$f^{\varepsilon}(\xi)\geq c_0'|\xi|^2; \tag{21}$$

$$f^{\varepsilon}(2\xi)\leq 2^{n+p}f^{\varepsilon}(\xi); \tag{22}$$

$$f^{\varepsilon}(\xi)\leq M\left(|\xi|+\frac{1}{4}\right)\qquad \textit{if } |\xi|\leq\frac{3}{4}; \tag{23}$$

$$\langle Df^{\varepsilon}(\xi),\lambda\rangle\leq pM|\lambda|\qquad \textit{if } |\xi|\leq\frac{3}{4}; \tag{24}$$

$$\langle Df^{\varepsilon}(\xi),\lambda\rangle\leq p\frac{f^{\varepsilon}(\xi)}{|\xi|-1/4}|\lambda|\qquad \textit{if } |\xi|\geq\frac{3}{4} \tag{25}$$

$$\langle D^2f^{\varepsilon}(\xi)\cdot\lambda,\lambda\rangle\geq 2\nu\frac{f^{\varepsilon}(\xi)}{|\xi|^2+1/16}|\lambda|^2, \tag{26}$$

where c_0' depends only on c_0.

Proof. By (4) we have

$$
\begin{aligned}
f^{\varepsilon}(\xi) &\geq c_0 \int_{B_1} \rho(\omega)|\xi + \varepsilon\omega|^2\, d\omega \\
&\geq c_0 \int_{(B_1 \setminus B_{1/2}) \cap \{\langle \xi, \omega\rangle \geq 0\}} \rho(\omega)[|\xi|^2 + \varepsilon^2|\omega|^2 + 2\varepsilon\langle \xi, \omega\rangle]\, d\omega \\
&\geq c_0\left(|\xi|^2 + \frac{\varepsilon^2}{4}\right)\int_{B_1 \setminus B_{1/2}} \rho(\omega)\, d\omega = c_0'\left(|\xi|^2 + \frac{\varepsilon^2}{4}\right),
\end{aligned}
$$

that is, (21).

By (9), for every $\xi \in \mathbb{R}^n$ we have that

$$
\begin{aligned}
f^{\varepsilon}(2\xi) &= \int_{B_1} \rho(\omega) f(2\xi + \varepsilon\omega)\, d\omega \\
&\leq 2^p \int_{B_1} \rho(\omega) f\left(\xi + \frac{\varepsilon}{2}\omega\right) d\omega \\
&= 2^{n+p} \int_{B_{1/2}} \rho(2\omega) f(\xi + \varepsilon\omega)\, d\omega \leq 2^{n+p} f^{\varepsilon}(\xi)
\end{aligned}
$$

since ρ decreases.

Now we observe that, by the convexity of f,

$$
f(\xi) \leq M|\xi|, \qquad \text{for every } |\xi| \leq 1 \tag{27}
$$

and by (6)

$$
\langle Df(\xi), \lambda\rangle \leq pM|\lambda| \qquad \text{for every } |\xi| \leq 1. \tag{28}
$$

Then, if $|\xi| \leq 3/4$, (23) follows by (27), while (28) leads to

$$
\langle Df^{\varepsilon}(\xi), \lambda\rangle \leq pM|\lambda|,
$$

that is, (24).

If $|\xi| \geq 3/4$, since $\varepsilon \leq 1/4$ we have by (6)

$$
\begin{aligned}
\langle Df^{\varepsilon}(\xi), \lambda\rangle &= \int_{B_1} \rho(\omega)\langle Df(\xi + \varepsilon\omega), \lambda\rangle\, d\omega \\
&\leq p\int_{B_1} \rho(\omega)\frac{f(\xi + \varepsilon\omega)}{|\xi| - \varepsilon|\omega|}|\lambda|\, d\omega \\
&\leq p\frac{f^{\varepsilon}(\xi)}{|\xi| - 1/4}|\lambda|.
\end{aligned}
$$

Furthermore it is easy to show again that (25), for $\lambda = \xi$, implies that

$$
f^{\varepsilon}(t\xi) \leq t^p f^{\varepsilon}(\xi) \qquad \forall\, t \geq 1. \tag{29}
$$

Finally, by (7) it follows that

$$\begin{aligned}\frac{1}{2}[f^\varepsilon(\xi)+f^\varepsilon(\lambda)] &= \frac{1}{2}\int_{B_1}\rho(\omega)[f(\xi+\varepsilon\omega)+f(\lambda+\varepsilon\omega)]\,d\omega \\ &\geq \int_{B_1}\rho(\omega)\bigg[f\left(\frac{\xi+\lambda}{2}+\varepsilon\omega\right) \\ &\quad + \nu\frac{|\xi-\lambda|^2}{|\xi+\varepsilon\omega|^2+|\lambda+\varepsilon\omega|^2}f\left(\frac{\xi+\lambda}{2}+\varepsilon\omega\right)\bigg]\,d\omega \\ &\geq f^\varepsilon\left(\frac{\xi+\lambda}{2}\right)+\int_{B_1}\frac{\rho(\omega)\nu|\xi-\lambda|^2 f\big((\xi+\lambda)/2+\varepsilon\omega\big)}{2|\xi|^2+2|\lambda|^2+4\varepsilon^2}\,d\omega.\end{aligned}$$

Then we have

$$\begin{aligned}\frac{1}{2}[f^\varepsilon(\xi)+f^\varepsilon(\lambda)] &\geq f^\varepsilon\left(\frac{\xi+\lambda}{2}\right) \\ &\quad + \frac{\nu|\xi-\lambda|^2}{2(|\xi|^2+|\lambda|^2+2\varepsilon^2)}\int_{B_1}\rho(\omega)f\left(\frac{\xi+\lambda}{2}+\varepsilon\omega\right)d\omega \\ &= f^\varepsilon\left(\frac{\xi+\lambda}{2}\right)+\frac{\nu|\xi-\lambda|^2}{2(|\xi|^2+|\lambda|^2+1/8)}f^\varepsilon\left(\frac{\xi+\lambda}{2}\right),\end{aligned}$$

thus (26) easily follows by applying Lemma 2.3 to f^ε. ∎

In the last result of this section, we show that convex functions of class Δ_2 for which condition (7) holds can be suitably approximated by means of regular convex functions (namely of class C^2) which satisfy conditions that are similar to (21) ... (26) and which behave as $|\xi|^2$ for large values of $|\xi|$. This result is a suitable modification of Lemma 2.4 of [11].

LEMMA 2.5. *Let $F\colon \mathbb{R}^n \to \mathbb{R}^+$ be a convex function satisfying conditions* (3) ... (7).

There exists a sequence $\{F_k\}_k$ of functions of class $C^2(\mathbb{R}^n)$ such that $F_k \to F$ uniformly on compact subsets of $\mathbb{R}^n$. Furthermore, for every $k \in \mathbb{N}$ we have

$$F_k(\xi) \geq c_1\left(|\xi|^2+\frac{1}{48k^2}\right), \tag{30}$$

where $c_1 > 0$ depends only on c_0,

$$F_k(2\xi) \leq \sigma_1 F_k(\xi), \tag{31}$$

with $\sigma_1 = \sigma_1(n, p) > 2$;

$$\langle DF_k(\xi), \lambda\rangle \leq pM|\lambda| \qquad \text{if } |\xi| \leq \frac{3}{4}, \tag{32}$$

$$\langle DF_k(\xi), \lambda\rangle \leq p_1\frac{F_k(\xi)}{|\xi|-1/2}c_2^{2^p-1} \qquad \text{if } |\xi| > \frac{3}{4}; \tag{33}$$

for every $\lambda \in \mathbb{R}^n$ *such that* $|\lambda| \le c_2$, *where* $c_2 > 1$ *is chosen suitably and* $p_1 = p_1(c_0, p, M) > 2$. *Finally*

$$\langle D^2 F_k(\xi) \cdot \lambda, \lambda \rangle \ge 2\nu_1 \frac{F_k(\xi)}{|\xi|^2 + 1/16k^2} |\lambda|^2, \tag{34}$$

with $\nu_1 = \nu_1(\nu, c_0, p) > 0$.

Proof. By means of Lemma 2.4, in the following we may assume that $F \in C^2(\mathbb{R}^n)$ satisfies assumptions (21)...(26).

Let us set

$$G_k(\xi) = (1 - \eta_k(\xi))F(\xi) + c_0' \eta_k(\xi)|\xi|^2. \tag{35}$$

The function η in (35) is assumed to be of class $C_0^1(\mathbb{R})$ and such that $\eta_k(\xi) = \eta(|\xi|/k)$, $0 \le \eta(t) \le 1$, $\eta(t) = 0$ if $t \le 1$, and $\eta(t) = 1$ if $t \ge 2$. Of course we have

$$c_0'|\xi|^2 \le G_k(\xi) \le F(\xi).$$

Let us denote by $G_k^{**}(\xi) = \sup\{G \le G_k\colon G \text{ is convex}\}$ the lower convex envelope of G_k.

For every $\xi, \lambda \in \mathbb{R}^n$ we have

$$c_0'\left(2\left|\frac{\xi + \lambda}{2}\right|^2 - |\xi|^2\right) \le c_0'|\lambda|^2 \le G_k(\lambda).$$

Since the left hand side of this inequality is a convex function of λ, by taking the supremum among all the convex functions below G_k and then letting $\lambda \to \xi$ we easily obtain that $G_k^{**}(\xi) \ge c_0'|\xi|^2$. Moreover, since when $|\xi| \ge 2k$ we have $G_k^{**}(\xi) \le G_k(\xi) = c_0'|\xi|^2$, it follows that

$$G_k^{**}(\xi) = c_0'|\xi|^2 \qquad \text{if } |\xi| \ge 2k.$$

Now we prove that there exists $k_0 > 1$ depending on c_0, p, M such that for every $|\xi| \le k^{1/2p}$ and every $k \ge k_0$

$$G_k^{**}(\xi) = F(\xi). \tag{36}$$

First we observe that for every $\xi \in \mathbb{R}^n$

$$G_k^{**}(\xi) \le G_k(\xi) \le F(\xi),$$

so that to prove (36) it is sufficient to prove that, for every $|\xi| \le k^{1/2p}$ and every $\lambda \in \mathbb{R}^n$,

$$\langle DF(\xi), \lambda - \xi \rangle + F(\xi) \le F(\lambda) = G_k(\lambda).$$

Namely, since F is convex, if $|\lambda| \le k$ we have

$$\langle DF(\xi), \lambda - \xi \rangle + F(\xi) \le F(\lambda) = G_k(\lambda).$$

Analogously, if $|\lambda| \geq k$ and $|\xi| \leq 3/4$, by (23) and (24) we have

$$\begin{aligned}\langle DF(\xi), \lambda - \xi\rangle + F(\xi) - G_k(\lambda) &\leq pM|\xi - \lambda| + M\left(|\xi| + \frac{1}{4}\right) - c_0'|\lambda|^2 \\ &\leq M(p + 1 + p|\lambda|) - c_0'k|\lambda| \\ &\leq Mp\left(\frac{p+1}{kp} + 1 - \frac{c_0'k}{Mp}\right)|\lambda|,\end{aligned}$$

and so there exists $k_1 = k_1(c_0, M, p) \geq 1$ such that for every $k \geq k_1$ (36) holds, provided

$$M\left(\frac{p+1}{k} + p\right) \leq c_0'k.$$

Finally, if $3/4 \leq |\xi| \leq k^{1/2p}$ and $|\lambda| \geq k$, by (25) and (29) we get

$$\begin{aligned}&\langle DF(\xi), \lambda - \xi\rangle + F(\xi) - G_k(\lambda) \\ &\quad\leq p\frac{F(\xi)}{|\xi| - \frac{1}{4}}|\xi - \lambda| + F(\xi) - G_k(\lambda) \\ &\quad\leq 2pF(\xi)(|\lambda| + |\xi|) + F(\xi) - G_k(\lambda) \\ &\quad\leq 2pF(\xi)\left(|\lambda| + \frac{k^{1/2p}}{k}|\lambda|\right) + F(\xi)\frac{|\lambda|}{k} - \frac{c_0'k}{Mk^{1/2}}F(\xi)|\lambda|\end{aligned}$$

and then it follows that (36) holds true provided $k_2 = k_2(c_0, p, M, L)$ is such that, for every $k \geq k_2$,

$$2p\left(1 + \frac{1}{k^{1-1/2p}}\right) + \frac{1}{k} \leq \frac{c_0'}{M}\sqrt{k}.$$

Then it is enough to set $k_0 = \max\{k_1, k_2\}$, to conclude this part of the proof.

Now let us define the functions

$$R_k(\xi) = \begin{cases} 0 & \text{if } \ |\xi| \leq \frac{k^{1/2p}}{2} \\ c_0'\left(|\xi|^2 - \frac{k^{1/p}}{4}\right) & \text{if } \ |\xi| \geq \frac{k^{1/2p}}{2}, \end{cases}$$

$$P_k(\xi) = G_k^{**}(\xi) + R_k(\xi)$$

and

$$F_k(\xi) = P_k^{1/4k}(\xi) = \int_{B_1} \rho(\omega)P_k\left(\xi + \frac{1}{4k}\omega\right) d\omega. \tag{37}$$

Since by (36), $P_k(\xi) = F(\xi)$ if $|\xi| \leq k^{1/2p}/2$, it follows that $F_k \to F$ uniformly on compact sets.

To conclude the proof we have to show that F_k satisfies (30) ...(34) for every k.

Since $R_k(\xi) \geq 0$, by (21) we have

$$\begin{aligned} F_k(\xi) &\geq c_0' \int_{B_1} \rho(\omega) \left| \xi + \frac{1}{4k}\omega \right|^2 d\omega \\ &\geq c_0' \int_{(B_1 \setminus B_{1/2}) \cap \{\langle \xi, \omega \rangle \geq 0\}} \rho(\omega) \left(|\xi|^2 + \frac{1}{16k^2}|\omega|^2 \right) d\omega \\ &\geq c_0' \left(|\xi|^2 + \frac{1}{48k^2} \right) \int_{B_1 \setminus B_{1/2}} \rho(\omega)\, d\omega \geq c_1 \left(|\xi|^2 + \frac{1}{48k^2} \right). \end{aligned}$$

For every $\xi \in \mathbb{R}^n$ we have

$$R_k(2\xi) \leq 4c_0'|\xi|^2 \leq 4G_k^{**}(\xi).$$

Furthermore, if ξ is such that $|\xi| \geq 2k$,

$$G_k^{**}(2\xi) = 4c_0'|\xi|^2 = 4G_k^{**}(\xi),$$

so that for every ξ such that $|\xi| \geq 2k$ we have

$$P_k(2\xi) \leq 8P_k(\xi).$$

Let now $|\xi| \leq k^{1/2p}/2$:

$$G_k^{**}(2\xi) = F(2\xi) \leq 2^p F(\xi) = 2^p G_k^{**}(\xi).$$

Finally, if $k^{1/2p}/2 \leq |\xi| \leq 2k$, let m be the smallest positive integer such that $m \geq 2k^{1-1/2p}$. Then $2m|\xi| \geq 2k$ and by the convexity of G_k^{**} we have

$$\begin{aligned} G_k^{**}(2\xi) &\leq \frac{1}{2} G_k^{**}(4\xi) \leq \cdots \leq \frac{1}{2^m} G_k^{**}(2m\xi) \\ &= \frac{c_0'}{2^m}|2m\xi|^2 \leq 4\frac{m^2}{2^m} G_k^{**}(\xi) \leq 4G_k^{**}(\xi), \end{aligned}$$

that is, for every $\xi \in \mathbb{R}^n$

$$G_k^{**}(2\xi) \leq \sigma_0 G_k^{**}(\xi), \tag{38}$$

where $\sigma_0 = 2^p$ $(= \max\{4, 2^p\})$. Then (31) easily follows by applying the same argument used to prove (22) in Lemma 2.4, thus obtaining that $\sigma_1 = 2^{n+p}$.

By means of (24) it is easy to see that if $|\xi| \leq 3/4$

$$|DF_k(\xi)| \leq pM.$$

For $3/4 \le |\xi| \le k^{1/2p} - 1/4k$, by (36), (25), and (21) we have

$$\begin{aligned}\langle DF_k(\xi), \lambda\rangle &\le p\int_{B_1} \rho(\omega)\frac{F(\xi+(1/4k)\omega)}{|\xi|-1/2}|\lambda|\,d\omega \\ &\quad + 2c_0'\int_{B_1}\rho(\omega)\left|\xi+\frac{1}{4k}\omega\right||\lambda|\,d\omega \\ &\le (p+2)\frac{F_k(\xi)}{|\xi|-1/2}|\lambda|.\end{aligned}$$

Furthermore, if $|\xi| \ge 2k + 1/4k$ we easily get

$$\langle DF_k(\xi), \lambda\rangle \le 4\frac{F_k(\xi)}{|\xi|-1/4}|\lambda|^2.$$

Finally, for $k^{1/2p} - 1/4k \le |\xi| \le 2k + 1/4k$, by (38) and (11) it follows that

$$\begin{aligned}\left\langle DG_k^{**}\left(\xi+\frac{1}{4k}\omega\right), \lambda\right\rangle &= \left\langle DG_k^{**}\left(\xi+\frac{1}{4k}\omega\right), \xi+\frac{1}{4k}\omega\right\rangle \\ &\quad + \left\langle DG_k^{**}\left(\xi+\frac{1}{4k}\omega\right), \lambda-\xi-\frac{1}{4k}\omega\right\rangle \\ &\le (2^p-2)G_k^{**}\left(\xi+\frac{1}{4k}\omega\right) + G_k^{**}(\lambda).\end{aligned}$$

By the way we will use these estimates in the next section, we may assume that there exists $c_2 > 9/8$ such that $|\lambda| \le c_2$. Then we have

$$\begin{aligned}G_k^{**}(\lambda) &\le M|\lambda|^{2^p-1}\frac{G_k^{**}(\xi+(1/4k)\omega)}{c_0'\left|\xi+(1/4k)\omega\right|^2} \\ &\le \frac{4Mc_2^{2^p-2}}{3c_0'}\frac{G_k^{**}(\xi+(1/4k)\omega)}{|\xi|-1/4}|\lambda|.\end{aligned}$$

Since G_k^{**} is convex, it follows that it satisfies a *Lipschitz condition* on the compact subsets of $\mathbb{R}^n$. In particular by (38) and (11) we may assume that $|G_k^{**}(x+y) - G_k^{**}(x)| \le (2^p-1)M|x|^{2^p-2}|y|$, for every $x, y \in E \subset\subset \mathbb{R}^n$.

By means of this property it is not difficult to show that, for every $\xi \in \mathbb{R}^n$ we have

$$F_k(\xi) \le F(\xi) + o\left(\frac{1}{k}\right). \tag{39}$$

Now, let us set $\zeta = 1/2k(\xi + (1/4k)\omega)$. Then $|\zeta| \le 9/8$ and by convexity it is easy to see that

$$G_k^{**}(\zeta) \le \frac{9}{8}\frac{G_k^{**}(\xi+1/4k\,\omega)}{|\xi|-1/4} \le \frac{G_k^{**}(\xi+1/4k\,\omega)}{|\xi|-1/4}c_2$$

and

$$\begin{aligned} G_k^{**}\left(\xi+\frac{1}{4k}\omega\right)-G_k^{**}(\zeta) &\le M(2^p-1)|\zeta|^{2p-2}\left(1-\frac{1}{2k}\right)\left|\xi+\frac{1}{4k}\omega\right| \\ &\le M(2^p-1)\left(\frac{9}{8}\right)^{2^p-2}\frac{G_k^{**}(\xi+(1/4k)\omega)}{c_0'(|\xi|-1/4)}c_2 \end{aligned}$$

and then, for every $\xi,\lambda\in\mathbb{R}^n$ such that $|\lambda|\le c_2$ and $k^{1/2p}-1/4k\le|\xi|\le 2k+1/4k$ we have

$$\begin{aligned} \left\langle DG_k^{**}\left(\xi+\frac{1}{4k}\omega\right),\lambda\right\rangle &\le (2^p-2)\left[G_k^{**}\left(\xi+\frac{1}{4k}\omega\right)-G_k^{**}(\zeta)\right] \\ &\quad+(2^p-2)G_k^{**}(\zeta)+G_k^{**}(\lambda) \\ &\le p_0\frac{G_k^{**}(\xi+(1/4k)\omega)}{|\xi|-1/4}c_2^{2^p-1}, \end{aligned}$$

where

$$p_0=p_0(c_0,p,M)=\frac{4M}{3c_0'}+\frac{M(2^p-2)(2^p-1)}{c_0'}+2^p-2.$$

By collecting all the previous estimates we obtain (33) with $p_1=p_0+2$.

Last we prove the uniform convexity of the sequence F_k, i.e., (34).

We split the space $\mathbb{R}^n$ in 3 regions, as before.

If $|\xi|\le k^{1/2p}/2-1/4k$ then $R_k(\xi+(1/4k)\omega)=0$, $G_k^{**}(\xi+(1/4k)\omega)=F(\xi+(1/4k)\omega)$ and so, by (26) we have

$$\langle D^2F_k(\xi)\lambda,\lambda\rangle\ge 4\nu\frac{F_k(\xi)}{|\xi|^2+1/16k^2}|\lambda|^2.$$

For every ξ such that $|\xi|\ge k^{1/2p}/2-1/4k$ we have

$$\left\langle D^2R_k\left(\xi+\frac{1}{4k}\omega\right)\lambda,\lambda\right\rangle=2c_0'|\lambda|^2\ge\frac{R_k(\xi+(1/4k)\omega)}{|\xi|^2+1/16k^2}|\lambda|^2.$$

Moreover, if $|\xi|\ge 2k+1/4k$,

$$\langle D^2F_k(\xi)\lambda,\lambda\rangle=4c_0'|\lambda|^2\ge\frac{F_k(\xi)}{|\xi|^2+1/16k^2}|\lambda|^2.$$

Finally, if $k^{1/2p}/2-1/4k\le|\xi|\le 2k+1/4k$ we have to show that

$$\left\langle D^2G_k^{**}\left(\xi+\frac{1}{4k}\omega\right)\lambda,\lambda\right\rangle\ge\nu_0\frac{G_k^{**}(\xi+(1/4k)\omega)}{|\xi|^2+1/16k^2}|\lambda|^2$$

for some $\nu_0>0$ independent of k.

By Lemma 2.3 this is equivalent to proving that, for every $\xi, \lambda \in \mathbb{R}^n$ such that $k^{1/2p}/2 - 1/4k \le |\xi|, |\lambda| \le 2k + 1/4k$,

$$\frac{1}{2}\left[G_k^{**}\left(\xi + \frac{1}{4k}\omega\right) + G_k^{**}\left(\lambda + \frac{1}{4k}\omega\right)\right] \ge G_k^{**}\left(\frac{\xi+\lambda}{2} + \frac{1}{4k}\omega\right) + \frac{\nu_0}{4}\frac{G_k^{**}((\xi+\lambda)/2 + (1/4k)\omega)}{|\xi|^2 + |\lambda|^2 + 1/16k^2}|\xi - \lambda|^2.$$

Fix $\zeta = 1/2k((\xi+\lambda)/2 + (1/4k)\omega)$, so that $|\zeta| \le 5/4$. By the convexity of G_k^{**} we have

$$G_k^{**}(\zeta) \le \frac{1}{4k}G_k^{**}\left(\xi + \frac{1}{4k}\omega\right) + \frac{1}{4k}G_k^{**}\left(\lambda + \frac{1}{4k}\omega\right)$$

and, since $G_k^{**}(\xi) \ge c_0'|\xi|^2$, by (38) we have

$$\begin{aligned} G_k^{**}\left(\frac{\xi+\lambda}{2} + \frac{1}{4k}\omega\right) - G_k^{**}(\zeta) &\le (2^p - 1)|\zeta|^{2^p-2}\left|\frac{\xi+\lambda}{2} + \frac{1}{4k}\omega - \zeta\right| \\ &\le (2^p-1)\left(1 + \frac{1}{4}\right)^{2^p-2}\frac{2k-1}{4k}\left(\left|\xi + \frac{1}{4k}\omega\right| + \left|\lambda + \frac{1}{4k}\omega\right|\right) \\ &\le c_k(c_0, p)\left[G_k^{**}\left(\xi + \frac{1}{4k}\omega\right) + G_k^{**}\left(\lambda + \frac{1}{4k}\omega\right)\right], \end{aligned}$$

where

$$c_k = \frac{2^p - 1}{c_0'}\left(\frac{5}{4}\right)^{2^p-2}\frac{2k-1}{4k^{1+1/2p} - 1}.$$

Then we have

$$\begin{aligned} &G_k^{**}\left(\frac{\xi+\lambda}{2} + \frac{1}{4k}\omega\right)\left[1 + \frac{\nu_0}{4}\frac{|\xi-\lambda|^2}{|\xi|^2 + |\lambda|^2 + 1/16k^2}\right] \\ &\quad\le \left(G_k^{**}\left(\frac{\xi+\lambda}{2} + \frac{1}{4k}\omega\right) - G_k^{**}(\zeta)\right)\left[1 + \frac{\nu_0}{4}\frac{|\xi-\lambda|^2}{|\xi|^2 + |\lambda|^2 + 1/16k^2}\right] \\ &\qquad + G_k^{**}(\zeta)\left[1 + \frac{\nu_0}{4}\frac{|\xi-\lambda|^2}{|\xi|^2 + |\lambda|^2 + 1/16k^2}\right] \\ &\quad\le \frac{1}{2}\left[G_k^{**}\left(\xi + \frac{1}{4k}\omega\right) + G_k^{**}\left(\lambda + \frac{1}{4k}\omega\right)\right] \end{aligned}$$

provided $\nu_0 > 0$ is such that for every $k \ge \tilde{k}(c_0, p) \ge 1$ large enough we have

$$\left(1 + \frac{\nu_0}{2}\right)\left[\frac{2^p - 1}{c_0'}\left(\frac{5}{4}\right)^{2^p-2}\frac{2k-1}{4k^{1+1/2p} - 1} + \frac{1}{4k}\right] \le \frac{1}{2}. \qquad \blacksquare$$

3. ESTIMATES FOR MINIMIZERS OF REGULAR FUNCTIONALS

In this section we deal with the functionals

$$\int_\Omega F_k(Dv)\,dx, \qquad v \in W^{1,F_k}_{\mathrm{loc}}(\Omega), \tag{40}$$

where F_k is the sequence approximating f, according to Lemma 2.5. The space W^{1,F_k} remains defined in an obvious way (see (2)).

We note that for minimizers v_k of functional (40) the usual *Euler equation*

$$\int_{\mathrm{supp}(\varphi)} \sum_{i=1}^n (D_{\xi_i} F_k(Dv_k)) D_i\varphi\,dx = 0 \tag{41}$$

holds for every $\varphi \in W^{1,F_k}_0(\Omega)$ such that $\mathrm{supp}(\varphi) \subset\subset \Omega$.

By means of the properties of functions F_k obtained in Lemma 2.5 we will prove the following

THEOREM 3.1. *Let $v_k \in W^{1,F_k}_{\mathrm{loc}}(\Omega)$ be a local minimizer of functional* (40), *where $F_k \in C^2(\mathbb{R}^n)$ satisfies* (30)–(34). *Then $v_k \in W^{2,2}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$. Moreover there exist two positive constants $C, \widetilde{C}$ which do not depend of k, such that the following estimates hold*

$$\int_{B_{R/2}} |D^2 v_k|^2\,dx \le C\, R^{-2(2^p-2)} \int_{B_R} (1 + F_k(Dv_k))\,dx; \tag{42}$$

$$\sup_{B_{R/2}} |Dv_k|^2 \le \frac{\widetilde{C}}{R^\mu} \int_{B_R} (1 + F_k(Dv_k))\,dx, \tag{43}$$

for some $\mu = \mu(n, p) > 0$ and every $R > 0$ such that $B_R \subset\subset \Omega$.

Proof. For our convenience in this section we will denote the functions F_k and the corresponding minimizers v_k respectively by F and v, emphasizing in the following the fact that the constants in the estimates that we will derive do not depend of the index k.

By our assumptions we may assume that v satisfies Eq. (41) and that conditions (32), (33) hold for the integrand function F.

Let $\psi \in C^1(\mathbb{R})$ be a non-decreasing odd function such that ψ is convex in $[0, +\infty)$ and $\psi'(t) \le c$ in $\mathbb{R}$ for some constant $c > 0$. Since $\psi(0) = 0$ we easily deduce that

$$|\psi(t)| \le t\psi'(t), \tag{44}$$

for every $t \in \mathbb{R}$. Moreover we fix $R > 0$ such that $B_R \subset\subset \Omega$. Let $\eta \in C^2_0(\Omega)$ be such that $0 \le \eta \le 1$, $\eta = 1$ in $B_{R/2}$, $\mathrm{supp}(\eta) \subset B_R$, $|D\eta| \le 2/R$,

$|D^2\eta| \le 4/R^2$. In particular we may assume, without loss of generality, that $R \le 1$ and $|D\eta| \le c_2 = 2/R$ (see the proof of Lemma 2.5 above).

Fix an integer $s \in \{1, \ldots, n\}$ and the corresponding coordinate unit vector e_s. We take into account Eq. (41), replacing f by F. If v is a function belonging to the space $W^{1,F}_{\mathrm{loc}}(\Omega)$, by Lemma 2.2 it follows that the function $\varphi = \Delta_{-h}(\eta^2\psi(\Delta_h v)) \in W^{1,F}_{\mathrm{loc}}(\Omega)$, for h sufficiently small. More precisely, by Lemma 2.2 there exists a number $\tau > 0$ such that $\tau\varphi \in W^{1,F}_{\mathrm{loc}}(\Omega)$ but, since we will use φ as a test function in Eq. (41), τ can be dropped.

If we put such a function in Eq. (41) and integrate by parts we have

$$\int_\Omega \sum_{i=1}^n \Delta_h(F_{\xi_i}(Dv))(\eta^2\psi'(\Delta_h v)\Delta_h(D_i v) + 2\eta D_i\eta\psi(\Delta_h v))\,dx = 0. \tag{45}$$

It is easy to see that

$$\Delta_h F_{\xi_i}(Dv) = \int_0^1 \sum_{j=1}^n F_{\xi_i\xi_j}(Dv + th\Delta_h(Dv))\,dt\Delta_h(D_j v) \tag{46}$$

$$= \int_0^1 \frac{d}{dx_s}(F_{\xi_i}(Dv(x + the_s)))\,dt. \tag{47}$$

Then, if we set $\xi_h^t = Dv + th\Delta_h(Dv) = tDv(x + he_s) + (1-t)Dv(x)$ and $\lambda_h^t = Dv(x + the_s)$, by using (46) in the first integral of formula (45) and (47) for the second one, after integrating by parts again the latter we have

$$\begin{aligned}\int_\Omega\int_0^1 \eta^2\psi'(\Delta_h v) \sum_{i,j=1}^n F_{\xi_i\xi_j}(\xi_h^t)\Delta_h(D_i v)\Delta_h(D_j v)\,dt\,dx \\ = 2\int_\Omega\int_0^1 \sum_{i=1}^n F_{\xi_i}(\lambda_h^t)[(D_s\eta D_i\eta + \eta D_s(D_i\eta))\psi(\Delta_h v) \\ + \eta D_i\eta\psi'(\Delta_h v)D_s(\Delta_h v)]\,dt\,dx.\end{aligned} \tag{48}$$

Moreover, by using (34) in (48) we get

$$\begin{aligned}2\nu_1\int_\Omega\int_0^1 \eta^2\psi'(\Delta_h v)\frac{F(\xi_h^t)}{|\xi_h^t|^2 + 1/16k^2}|\Delta_h(Dv)|^2\,dt\,dx \\ \le \int_\Omega\int_0^1 \psi(\Delta_h v)\langle DF(\lambda_h^t), D(D_s(\eta^2))\rangle\,dt\,dx \\ + 2\int_\Omega\int_0^1 \left(\eta^2\psi'(\Delta_h v)\frac{F(\lambda_h^t)}{|\lambda_h^t|^2 + 1/16k^2}|\Delta_h(Dv)|^2\right)^{1/2} \\ \cdot\left(\psi'(\Delta_h v)|\langle DF(\lambda_h^t), D\eta\rangle|^2\frac{|\lambda_h^t|^2 + 1/16k^2}{F(\lambda_h^t)}\right)^{1/2}dt\,dx.\end{aligned} \tag{49}$$

By (44) and inequality $ab \le \varepsilon a^2 + b^2/4\varepsilon$, where we set $\varepsilon = \nu_1$, (49) leads to

$$\begin{aligned}
2\nu_1 & \int_\Omega \int_0^1 \eta^2 \psi'(\Delta_h v) \frac{F(\xi_h^t)}{|\xi_h^t|^2 + 1/16k^2} |\Delta_h(Dv)|^2 \, dt\, dx \\
& - \nu_1 \int_\Omega \int_0^1 \eta^2 \psi'(\Delta_h v) \frac{F(\lambda_h^t)}{|\lambda_h^t|^2 + 1/16k^2} |\Delta_h(Dv)|^2 \, dt\, dx \\
& \le \int_\Omega \int_0^1 \psi'(\Delta_h v) |\Delta_h v| |\langle DF(\lambda_h^t), D(D_s(\eta^2))\rangle| \, dt\, dx \\
& + \frac{1}{\nu_1} \int_\Omega \int_0^1 \psi'(\Delta_h v) |\langle DF(\lambda_h^t), D\eta\rangle|^2 \frac{|\lambda_h^t|^2 + 1/16k^2}{F(\lambda_h^t)} \, dt\, dx
\end{aligned}$$

and so

$$\begin{aligned}
2\nu_1 & \int_{B_R} \int_0^1 \eta^2 \psi'(\Delta_h v) \frac{F(\xi_h^t)}{|\xi_h^t|^2 + 1/16k^2} |\Delta_h(Dv)|^2 \, dt\, dx \\
& - \nu_1 \int_{B_R} \int_0^1 \eta^2 \psi'(\Delta_h v) \frac{F(\lambda_h^t)}{|\lambda_h^t|^2 + 1/16k^2} |\Delta_h(Dv)|^2 \, dt\, dx \\
& \le \int_{B_R} \int_0^1 \psi'(\Delta_h v) |\langle DF(\lambda_h^t), D(D_s(\eta^2))\rangle| \; |\Delta_h v| \; dt\, dx \\
& + \frac{1}{\nu_1} \int_{B_R} \int_0^1 \psi'(\Delta_h v) \frac{|\lambda_h^t|^2 + 1/16k^2}{F(\lambda_h^t)} \; |\langle DF(\lambda_h^t), D\eta\rangle|^2 \, dt\, dx \\
& \le \sum_{i=1}^4 \mathcal{I}_i, \qquad (50)
\end{aligned}$$

where the integrals $\mathcal{I}_i$, $i = 1, \ldots, 4$, are defined below. First of all let us assume for now that $\psi(t) = t$, that is, $\psi'(t) = 1$. Then for every $k = 1, 2, \ldots,$ by (30) ... (33) we have

$$\begin{aligned}
I_1 &= \int_{B_R} \int_0^1 \chi_{\{|\lambda_h^t| \le 3/4\}} |\langle DF(\lambda_h^t), D(D_s(\eta^2))\rangle| |\Delta_h v| \, dt\, dx \\
&\le \frac{4pM}{R^2} \int_{B_R} \int_0^1 |\Delta_h v| \, dt\, dx \le \frac{4pM}{R^2} \|Dv\|_{L^2(B_R)} |B_R|^{1/2} < +\infty; \qquad (51)
\end{aligned}$$

by setting $\beta = 2^p - 1$ we have

$$\begin{aligned}
I_2 &= \int_{B_R} \int_0^1 \chi_{\{3/4 \le |\lambda_h^t| \le 2k + 1/4k\}} |\langle DF(\lambda_h^t), D(D_s(\eta^2))\rangle| |\Delta_h v| \, dt\, dx \\
& + \int_{B_R} \int_0^1 \chi_{\{|\lambda_h^t| \ge 2k + 1/4k\}} |\langle DF(\lambda_h^t), D(D_s(\eta^2))\rangle| |\Delta_h v| \, dt\, dx \\
&\le \frac{4^\beta p_1 M}{R^{2\beta}} \int_{B_R} \int_0^1 \chi_{\{3/4 \le |\lambda_h^t| \le 2k + 1/4k\}} \frac{|\lambda_h^t|^{p_1}}{|\lambda_h^t| - 1/2} |\Delta_h v| \, dt\, dx
\end{aligned}$$

$$
\begin{aligned}
&+\frac{4^{\beta+1}p_1c_0'}{R^{2\beta}}\int_{B_R}\int_0^1\chi_{\{|\lambda_h^t|\ge 2k+1/4k\}}\frac{|\lambda_h^t|^2+1/16k^2}{|\lambda_h^t|-1/2}|\Delta_h v|\,dt\,dx\\
&\le\frac{2p_14^\beta M}{R^{2\beta}}\left(2k+\frac{1}{4k}\right)^{p_1}\|Dv\|_{L^2(B_R)}|B_R|^{1/2}\\
&\quad+\frac{4^\beta p_1c_0'}{kR^{2\beta}}\left(8k^2+1+\frac{1+4k^2}{8k^2+1-2k}\right)\|Dv\|_{L^2(B_R)}|B_R|^{1/2}<+\infty;
\end{aligned}
$$

$$
\begin{aligned}
I_3&=\frac{1}{\nu_1}\int_{B_R}\int_0^1\chi_{\{|\lambda_h^t|\le 3/4\}}\frac{|\lambda_h^t|^2+1/16k^2}{F(\lambda_h^t)}|\langle DF(\lambda_h^t),D\eta\rangle|^2\,dt\,dx\\
&\le\frac{4p^2M^2}{\nu_1c_1R^2}\int_{B_R}\int_0^1\chi_{\{|\lambda_h^t|\le 3/4\}}\frac{|\lambda_h^t|^2+1/16k^2}{|\lambda_h^t|^2+1/48k^2}\,dt\,dx\\
&\le\frac{12p^2M^2}{\nu_1c_1R^2}|B_R|<+\infty,
\end{aligned}
$$

and finally

$$
\begin{aligned}
I_4&=\frac{1}{\nu_1}\int_{B_R}\int_0^1\chi_{\{|\lambda_h^t|\ge 3/4\}}\frac{|\lambda_h^t|^2+1/16k^2}{F(\lambda_h^t)}\;|\langle DF(\lambda_h^t),D\eta\rangle|^2\,dt\,dx\\
&\le\frac{4^\beta p_1^2}{\nu_1R^{2\beta}}\int_{B_R}\int_0^1\chi_{\{|\lambda_h^t|\ge 3/4\}}\frac{|\lambda_h^t|^2+1/16k^2}{\left(|\lambda_h^t|-1/2\right)^2}F(\lambda_h^t)\,dt\,dx\\
&\le\frac{10p_1^24^\beta}{\nu_1R^{2\beta}}\int_{B_R}F(Dv)\,dx<+\infty.
\end{aligned}
$$

Thus, since $R\le 1$, by (30) (32), (33), and letting $h\to 0$ in (50) we easily get, for every $s=1,\dots,n$,

$$
\int_{B_{R/2}}|D_s(Dv)|^2\,dx\le\frac{c_3}{R^{2\beta}}\int_{B_R}(1+F(Dv))\,dx<+\infty, \tag{52}
$$

where $c_3=c_3(n,p,c_0,M)$ and $\beta=2(2^p-1)$. Thus by (52) it follows (42).

Now we turn our attention again to inequality (50), assuming ψ' to not be constant. Since ψ is Lipschitz-continuous, the right hand side of (50) can be estimated as we made for $I_1,\dots,I_4$ above, as $\Delta_h u$ converges to $D_s u$ a.e. Hence by the dominate convergence theorem we get

$$
\begin{aligned}
&\int_{B_R}\eta^2\psi'(D_sv)\frac{F(Dv)}{|Dv|^2+1/16k^2}|D_s(Dv)|^2\,dx\\
&\qquad\le\frac{c_4}{R^{2\beta}}\int_{B_R}(1+F(Dv)\psi'(D_sv))\,dx.
\end{aligned}\tag{53}
$$

It is also remarkable that inequality (53) holds true also for a more general ψ; namely if ψ' is not bounded we can approximate ψ by means of a sequence ψ_r such that $\psi_r = \psi$ in $[-r, r]$ for every $r > 0$, while ψ_r is extended linearly outside this interval, as a function of class C^1 in $\mathbb{R}$. If we insert ψ_r in the place of ψ in (50), we can easily check that, by means of the dominate convergence theorem, (53) still holds as we let $r \to +\infty$.

For every $t > 0$, let us introduce the function

$$\Phi(t) = 1 + \int_0^t \sqrt{\psi'(s) \frac{F((1 + s/|Dv|)Dv)}{(1 + s/|Dv|)^2(|Dv|^2 + 1/16k^2)}}\, ds.$$

It is easy to prove that

$$|D(\eta\Phi(|D_s v|))|^2 \leq 2|D\eta|^2(\Phi(|D_s v|))^2 + 2\sigma_1\eta^2\psi'(|D_s v|)\frac{F(Dv)}{|Dv|^2 + 1/16k^2}|D_s(Dv)|^2 \tag{54}$$

and, by means of (31) and (44)

$$(\Phi(|Dv|))^2 \leq 2 + 2\sigma_1 F(Dv)\psi'(|D_s v|). \tag{55}$$

Then by (53), (54), (55), the Sobolev inequality, and recalling that since ψ is odd, ψ' turns out to be even, we have

$$\left(\int_{B_{R/2}} [\Phi(|D_s v|)]^{2^*}\, dx\right)^{2/2^*} \leq \frac{c_5}{R^{2\beta}} \int_{B_R} [1 + F(Dv)\psi'(|D_s v|)]\, dx. \tag{56}$$

Now for $\alpha \geq 0$ and every $t \geq 0$ we set $\psi'(t) = t^{2\alpha}$. Then

$$\begin{aligned}
[\Phi(|D_s v|)]^{2^*} &\geq 1 + \frac{[F(Dv)]^{2^*/2}}{[4(|Dv|^2 + 1/64k^2)]^{2^*/2}}\left(\int_0^{|D_s v|} s^\alpha\right)^{2^*} \\
&\geq 1 + (c_1)^{2^*/2-1}\frac{F(Dv)}{|Dv|^2}\left[\frac{|Dv|^2}{4(|Dv|^2 + 1/64k^2)}\right]^{2^*/2}\frac{|D_s v|^{2^*(\alpha+1)}}{(\alpha+1)^{2^*}} \\
&\geq \frac{c_6}{(\alpha+1)^{2^*}}\left[1 + \frac{F(Dv)}{|Dv|^2}\left[\frac{|Dv|^2}{|Dv|^2 + 1/64k^2}\right]^{2^*/2}|D_s v|^{2^*(\alpha+1)}\right],
\end{aligned}$$

where

$$c_6 = \min\left\{1, \frac{1}{4}\left(\frac{c_1}{4}\right)^{2^*/2-1}\right\}.$$

Then if $|Dv| \geq 1$ it follows that

$$[\Phi(|D_s v|)]^{2^*} \geq \frac{c_6}{2(\alpha+1)^{2^*}}\left[1 + \frac{F(Dv)}{|Dv|^2}|D_s v|^{2^*(\alpha+1)}\right].$$

On the other hand, if $|Dv| \le 1$, since $\Phi(t) \ge 1$ we have

$$\frac{c_6}{2(\alpha+1)^{2^*}}\left[1+\frac{F(Dv)}{|Dv|^2}|D_s v|^{2^*(\alpha+1)}\right] \le \frac{c_6}{2}\left[1+M|Dv|^{2^*(\alpha+1)-2}\right]$$
$$\le \frac{c_6(1+M)}{2}[\Phi(|D_s v|)]^{2^*}$$

and finally

$$[\Phi(|D_s v|)]^{2^*} \ge \min\left\{\frac{1}{M+1}, \frac{c_6}{2}\right\}\frac{1}{(\alpha+1)^{2^*}}\left[1+\frac{F(Dv)}{|Dv|^2}|D_s v|^{2^*(\alpha+1)}\right]$$

for every value of $|Dv|$. Then, by using the last inequality in (56) and adding up over s, we easily obtain

$$\left(\int_{B_{R/2}}[1+F(Dv)|Dv|^{2^*(\alpha+1)-2}]\,dx\right)^{2/2^*}$$
$$\le \frac{c_7(\alpha+1)^2}{R^{2\beta}}\int_{B_R}[1+F(Dv)|Dv|^{2\alpha}]\,dx. \tag{57}$$

Now we introduce the following sequence of numbers

$$\alpha_0 = 0, \qquad \alpha_{i+1} = \frac{2^*}{2}(\alpha_i+1)-1 \quad \forall\, i \ge 0,$$

and a sequence of radii

$$R_i = \frac{R}{2}+\frac{R}{2^{i+1}}.$$

Of course $\alpha_i \nearrow +\infty$, while $R_0 = R$ and $R_i \searrow R/2$. Moreover $R_i - R_{i+1} = R/2^{i+2}$.

It is immediate to observe that

$$\alpha_i = \left(\frac{2^*}{2}\right)^i - 1 \tag{58}$$

for every $i \ge 0$.

Now let us put α_i in the place of α and R_i, R_{i+1}, respectively, in the place of R and $\frac{R}{2}$ in (57). Then we obtain

$$\left(\int_{B_{R_{i+1}}}\left[1+\frac{F(Dv)}{|Dv|^2}|Dv|^{2(\alpha_{i+1}+1)}\right]dx\right)^{2/2^*}$$
$$\le \frac{c_7 4^{i+2}(\alpha_i+1)^2}{R^{2\beta}}\int_{B_{R_i}}\left[1+\frac{F(Dv)}{|Dv|^2}|Dv|^{2(\alpha_i+1)}\right]dx. \tag{59}$$

We set

$$E_i = \left(\int_{B_{R_i}} \left[1 + \frac{F(Dv)}{|Dv|^2} |Dv|^{2(\alpha_i+1)} \right] dx \right)^{1/\alpha_i+1}.$$

Then by iterating inequality (59) we obtain

$$\begin{aligned} E_{i+1} &\le \left[\frac{c_7 4^{i+2}(\alpha_i+1)^2}{R^{2\beta}} \right]^{1/\alpha_i+1} E_i \\ &\le \prod_{j=0}^{i} \left[\frac{c_7 4^{j+2}(\alpha_j+1)^2}{R^{2\beta}} \right]^{1/\alpha_j+1} E_0 \le \frac{c_8}{R^\mu} E_0, \end{aligned} \tag{60}$$

where as we can easily check,

$$c_8 = \exp\left[\sum_{j=0}^{\infty} \frac{1}{\alpha_j+1} \log\left(c_7 4^{j+2}(\alpha_j+1)^2 \right) \right] < +\infty.$$

Furthermore, by (58) it follows that

$$\mu = \mu(n, p) = \sum_{i=0}^{\infty} \frac{2\beta}{\alpha_i+1} = \frac{2\beta 2^*}{2^*-2}$$

(and in particular, if $n > 2$, $\mu = n(2^p - 2)$).

Finally, by (4) and by letting $i \to +\infty$ in (60) we obtain

$$\begin{aligned} \sup_{B_{R/2}} |Dv|^2 &= \lim_{i\to+\infty} \left(\int_{B_{R/2}} |Dv|^{2\alpha_{i+1}} \, dx \right)^{1/\alpha_{i+1}} \\ &\le \frac{1}{c_1} \lim_{i\to+\infty} E_{i+1} \le \frac{c_8}{c_1 R^\mu} \int_{B_R} (1 + F(Dv)) \, dx < +\infty \end{aligned}$$

and this inequality concludes the proof of Theorem 3.1. ∎

4. PROOF OF THEOREM 1.1

In this section we prove the statement of Theorem 1.1, by means of the results obtained in Sections 2 and 3.

Namely let $u \in W^{1,f}_{\mathrm{loc}}(\Omega)$ be a local minimizer of functional (1) and $R > 0$ such that $B_R \subset\subset \Omega$. We consider the family of Dirichlet problems

$$\min\left\{ \int_{B_R} F_k(Dw) \, dx \colon w \in u + W^{1,F_k}(B_R) \right\}. \tag{61}$$

For every k, let v_k be a solution to such a problem.

By (43) and (39), for every $k \in \mathbb{N}$ we have

$$\frac{1}{\widetilde{C}} \sup_{B_{R/2}} |Dv_k|^2 \le \int_{B_R} (1 + F_k(Dv_k))\, dx \le \int_{B_R} (1 + F_k(Dv))\, dx$$
$$\le \int_{B_R} (1 + f(Du))\, dx + o\left(\frac{1}{k}\right)|B_R| < +\infty.$$

Then, up to a subsequence, $Dv_k \rightharpoonup Dw$ in the weak-* topology of L^∞, where $w = u$ on ∂B_R. Furthermore for every $\rho < R$ we have

$$\sup_k \left(\sup_{B_\rho} |Dv_k|^2 \right) < +\infty,$$

and then

$$\int_{B_\rho} f(Dw)dx \le \liminf_k \int_{B_\rho} f(Dv_k)dx$$
$$= \liminf_k \left[\int_{B_\rho} F_k(Dv_k)dx + \int_{B_\rho} (f(Dv_k) - F_k(Dv_k))dx \right]. \quad (62)$$

Now since $F_k \to f$ uniformly on compact sets and since

$$\sup_{B_\rho} |Dv_k|^2 \le c(\rho)$$

for every k, we deduce that

$$\liminf_k \int_{B_\rho} (f(Dv_k) - F_k(Dv_k))\, dx = 0.$$

Finally, by (62) and (39) we have that

$$\int_{B_\rho} f(Dw)\, dx \le \liminf_k \int_{B_\rho} F_k(Dv_k)\, dx$$
$$\le \liminf_k \int_{B_R} F_k(Dv_k)\, dx \le \int_{B_R} f(Du)\, dx,$$

so that, by letting $\rho \to R$ we conclude the proof by the minimality of u for functional (1). ∎

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